國立高雄師範大學 107 學年度學士班轉學生招生考試試題

系所別：數學系三年級

科　目：高等微積分

※注意：1. 不必抄題，作答時請將試題題號及答案依照順序寫在答案卷上，於本試題上作答者，不予計分。

2. 限用藍色或黑色之鋼筆、原子筆作答，以鉛筆或其他顏色作答者不予計分。

1. Prove that a^x is analysis on $\mathbb{R}$ for each $a > 0$.（9%）

2. Fix $T \in \mathcal{L}(\mathbb{R}^n; \mathbb{R}^m)$. Set

$$M_1 \equiv \sup_{\|x\|=1} \|T(x)\| \text{ and}$$

$$M_2 \equiv inf\{C > 0 : \|T(x)\| \le C\|x\| \; for\ all\ x \in \mathbb{R}^n\}$$

   a) Prove that $M_1 \le \|T\|$.（3%）

   b) Using the linear property of T, prove that if $x \neq 0$ then $\frac{\|T(x)\|}{\|x\|} \le M_1$.（3%）

   c) Prove that $M_1 = M_2 = \|T\|$.（3%）

3. Suppose that E is a bounded noncompact subset of $\mathbb{R}^n$ and that $f: E \longrightarrow (0, \infty)$. If there is a function $g: E \longrightarrow \mathbb{R}$ such that $g(x) > f(x)$ for all $x \in E$, then prove that there exist $x_1, \cdots, x_N \in E$ such that

$$E \subset \bigcup_{j=1}^{N} B_{g(x_j)}(x_j)$$

（9%）

4. Define a function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ by

$$f(x,y) = \begin{cases} \dfrac{x^3 - xy^2}{x^2 + y^2}, & if \quad (x,y) \neq (0,0) \\ 0, & if \quad (x,y) = (0,0) \end{cases}$$

   Show that f is continuous and has all directional derivatives in all directions at $(0,0)$, but is not differentiable there.（9%）

（背面有題　續翻背面）

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5. (a) Consider the sequence of functions $f_k(x) = \frac{1}{k}\sin x$, where $k \in \mathbb{N}$. Show that $\{f_k\}$ converges to zero uniformly on $\mathbb{R}$.（4%）

   (b) Consider the sequence of functions $g_k(x) = \frac{1}{k}x\sin x$, where $k \in \mathbb{N}$. Show that $\{g_k\}$ converges pointwise, but not uniformly, to zero on $\mathbb{R}$.（5%）

6. Let $x_n = \sin\frac{n\pi}{2}$, where $n \in \mathbb{N}$. Find $\inf_{k\geq 5} x_{2k}$, $\sup_{k\geq 5} x_{3k}$, $\liminf_{n\to\infty} x_n$.（9%）

7. Let $x_n = \frac{1}{n^2}$, where $n \in \mathbb{N}$. Check that $\{x_n\}$ is a Cauchy sequence.（9%）

8. (a) Let $f_k(x) = \sin(kx^2)$, where $k \in \mathbb{N}$. Does the sequence $\{f_k\}$ converge pointwise on $\mathbb{R}$? Does it converge uniformly on $\mathbb{R}$? Justify your answer.（5%）

   (b) Let $f_k(x) = \sin\left(\frac{1}{k}x^2\right)$, where $k \in \mathbb{N}$. Does the sequence $\{f_k\}$ converge pointwise on $\mathbb{R}$? Does it converge uniformly on $\mathbb{R}$? Justify your answer.（5%）

9. Find a function $f\colon \mathbb{R} \to \mathbb{R}$ such that $F(x) := \int_{-\infty}^{x} f(t)dt$ exists for all x, but not differentiable at $x = 0$. Justify your answer.（9%）

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10. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a convex function, i.e., for all $t \in (0,1)$ and

$(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$,

$$tf(x_1, y_1) + (1-t)f(x_2, y_2) \geq f(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2)$$

Must $\Omega := \{(x,y) \in \mathbb{R}^2 | f(x,y) \leq c\}$ be a convex set in $\mathbb{R}^2$ for any given $c \in \mathbb{R}$? Justify your answer.（9%）

11. Consider a sequence of differentiable functions $f_n: \mathbb{R} \to \mathbb{R}$. Must the limit function $f := \lim_{n\to\infty} f_n$ be differentiable? Prove it or construct an explicit counter-example.（9%）